§9.5 Linear First Order Differential Equations

HW §9.5 # 1-20

A first-order differential equation is one that can be put in the form

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

where $P(x)$ and $Q(x)$ are continuous functions on a given interval.

Example: Solve $xy' + y = 2x$

Step 0: Divide through by the coefficient of y' if necessary.

$$y' + \frac{1}{x}y = 2$$

Note that $(xy)' = xy' + 1 \cdot y$

~~To sl~~ To solve, it would be better to have this in the form

$$xy' + y = 2x$$

$$(xy)' = 2x$$

Integrate both sides

$$\int (xy)' \, dx = \int 2x \, dx$$

$$xy = \frac{\cancel{2}x^2}{\cancel{2}} + C$$

$$y = \frac{x^2 + C}{x}$$

$$y = \frac{x^2}{x} + \frac{C}{x}$$

$$y = x + \frac{C}{x} \quad \square$$

All first order linear DE can be solved in a similar way by multiplying the equation by what is called an <u>integrating</u> <u>factor</u>, $I(x)$

We start with

$$y' + P(x)y = Q(x)$$

We want a function $I(x)$ such that

$$I(x)\left[y' + P(x)y\right] = (I(x)y)'$$

so that when we multiply $y' + P(x)y = Q(x)$ through by $I(x)$, we get

$$I(x)(y' + P(x)y) = I(x)Q(x)$$

The left hand side will become $(I(x)y)'$

$$(I(x)y)' = I(x)Q(x)$$

integrate both side

$$\int (I(x)y)'dx = \int I(x)Q(x)dx$$

$$I(x)y = \int I(x)Q(x)\cancel{dx} + C$$

$$\boxed{y = \frac{\int I(x)Q(x)dx + C}{I(x)}}$$

Let's get a formula for $I(x)$, we want

$$I(x)[y' + P(x)y] = (I(x)y)'$$

product rule

$$\cancel{I(x)y'} + I(x)P(x)y = I'(x)y + \cancel{I(x)y'}$$

$$I(x)P(x)y = I'(x)y$$

so $\quad I(x)P(x) = I'(x)$

$$I\,P(x) = \frac{dI}{dx}$$

This is a separable DE $\longrightarrow$

$$\frac{dI}{dx} = I\,P(x)$$

$$\frac{dI}{I} = P(x)dx$$

$$\int \frac{dI}{I} = \int P(x)dx$$

$\ln|I| = \int P(x)\,dx + C$

Apply exp function to both sides

$e^{\cancel{\ln}|I|} = e^{\int P(x)\,dx + C}$

$|I| = e^{\int P(x)\,dx}$

• we don't need absolute value • choose $C=0$

$$\boxed{I = e^{\int P(x)\,dx}}$$

and

$$\boxed{y = \frac{\int I(x)\,Q(x)\,dx + C}{I(x)}}$$

where

$y' + P(x)\,y = Q(x)$

Example: Solve $y' + 3x^2 y = 6x^2$

SOLUTION:

$$P(x) = 3x^2$$
$$Q(x) = 6x^2$$
$$I = e^{\int P(x)\,dx}$$
$$= e^{\int 3x^2\,dx}$$
$$= e^{3x^3/3} = e^{x^3}$$

We can now use the formula, or we can do a more hands approach. Multiply through by $I = e^{x^3}$

$$e^{x^3} y' + 3x^2 e^{x^3} y = 6x^2 e^{x^3}$$
$$(e^{x^3} y)' = 6x^2 e^{x^3}$$
$$e^{x^3} y = \int \overset{(A)}{6} x^2 e^{x^3}\,dx + C$$

(A) $\int 6x^2 e^{x^3}\,dx = \int 6 \cdot \frac{1}{3} e^u\,du = 2e^u$
$= 2e^{x^3}$

$u = x^3$
$du = 3x^2\,dx$
$\frac{1}{3}du = x^2\,dx$

$$e^{x^3} y = 2e^{x^3} + C$$
$$y = \frac{2e^{x^3} + C}{e^{x^3}}$$

$$y = \frac{2e^{x^3}}{e^{x^3}} + \frac{C}{e^{x^3}}$$

$$\boxed{y = 2 + Ce^{-x^3}}$$

Example: Solve the initial value problem

$$x^2y' + xy = 1, \quad x > 0, \ y(1) = 2$$

SOLUTION Divide through by x^2

$$(\cancel{*}) \ y' + \frac{x}{x^2}y = \frac{1}{x^2}$$

$$(*) \ y' + \frac{1}{x}y = \frac{1}{x^2}$$

$$P(x) = \frac{1}{x}$$

$$I = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx}$$

$$= e^{\ln x}$$

$$I = x$$

Multiply through $(*)$ by $I = x$

$$xy' + y = \frac{1}{x}$$

$$(xy)' = \frac{1}{x}$$

$$xy = \int \frac{1}{x}dx + C$$

$$xy = \ln(x) + C$$

$$y = \frac{\ln(x) + C}{x}$$

Solve for c.

~~when~~ we have $y(1) = 2$.

so when $x = 1$, $y = 2$.

$$2 = \frac{\underbrace{\ln(1)}_{=0} + c}{1}$$

$$2 = c$$

$$c = 2$$

$$\boxed{y = \frac{\ln(x) + 2}{x}}$$

Aside Simplify

$$e^{\int \frac{2}{x}\,dx}, \qquad x > 0$$

$$= e^{2 \ln x}$$

$$= e^{\ln x^2}$$

$$= x^2$$

§8.2 #9 Find the area of the surface obtained by rotating the curve about the x-axis.

$$y = \sin \pi x, \quad 0 \le x \le 1$$

$$S = \int 2\pi y \, ds$$

$$ds = \sqrt{1+(y')^2}\, dx$$

$$y' = \pi \cos \pi x$$

$$S = \int_0^1 2\pi \sin \pi x \sqrt{1+(\pi\cos\pi x)^2}\, dx$$

$$u = \pi \cos \pi x$$

$$du = -\pi^2 \sin \pi x \, dx$$

$$-\frac{1}{\pi^2} du = \sin \pi x \, dx$$

- $x = 0$
  $u = \pi \cos(0)$
  $= \pi$
- $x = 1$
  $u = \pi \cos \pi \cdot 1$
  $= \pi(-1)$
  $u = -\pi$

$$= \int_{\pi}^{-\pi} 2\pi \cdot \left(\frac{-1}{\pi^2}\right) \sqrt{1+u^2}\, du$$

$$= -\frac{2}{\pi} \int_{\pi}^{-\pi} \sqrt{1+u^2}\, du$$

$$= \frac{2}{\pi} \int_{-\pi}^{\pi} \underbrace{\sqrt{1+u^2}}_{\text{even}}\, du$$

$$= \frac{2 \cdot 2}{\pi} \int_0^{\pi} \sqrt{1+u^2}\, du$$

$$= \frac{4}{\pi} \int_0^{\pi} \sqrt{1+u^2}\, du$$

$u = \tan\theta$
$du = \sec^2\theta \, d\theta$

$$= \frac{4}{\pi}\int_0^{\alpha} \sqrt{1+\tan^2\theta} \cdot \sec^2\theta \, d\theta$$

- $u = 0$
  $\tan\theta = 0$
  $\theta = 0$
- $u = \pi$
  $\tan\theta = \pi$
  $\theta = \tan^{-1}(\pi)$
  $\alpha = \tan^{-1}(\pi)$

$$= \frac{4}{\pi}\int_0^{\alpha} \sqrt{\sec^2\theta} \cdot \sec^2\theta \, d\theta$$

$$= \frac{4}{\pi}\int_0^{\alpha} \sec\theta \cdot \sec^2\theta \, d\theta$$

$$= \frac{4}{\pi}\int_0^{\alpha} \sec^3\theta \, d\theta$$

$n = 3$

Reduction FORMULA

$$\int \sec^n x \, dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x \, dx$$

$$= \frac{4}{\pi}\left[\frac{\tan x \sec x}{2} + \frac{1}{2}\int \sec x \, dx\right]$$

$$= \frac{4}{\pi}\left[\frac{\tan\theta \sec\theta}{2} + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right]_0^{\alpha}$$

$$= \frac{4}{\pi}\left[\frac{\tan\alpha \sec\alpha}{2} + \frac{1}{2}\ln|\sec\alpha + \tan\alpha|\right]$$

$$- \frac{4}{\pi}\left[\frac{\overset{=0}{\tan 0}\sec 0}{2} + \frac{1}{2}\ln|\underset{1}{\sec 0} + \underset{0}{\tan 0}|\right]$$

$\ln 1 = 0$

~~tan~~ $\alpha = \tan^{-1}(\pi)$

$\tan\alpha = \pi$

$1 + \tan^2\alpha = \sec^2\alpha$

$1 + \pi^2 = \sec^2\alpha$

$\sec\alpha = \sqrt{1+\pi^2}$

$$= \frac{4}{\pi}\left[\frac{1}{2}\pi\sqrt{1+\pi^2} + \frac{1}{2}\ln\left|\sqrt{1+\pi^2}+\pi\right|\right]$$

$$= 2\sqrt{1+\pi^2} + \frac{2}{\pi}\ln\left|\sqrt{1+\pi^2}+\pi\right|$$