2.6 The Inverse of a Square Matrix

Homework §2.6 #1-21 odd.

Theorem 1. *Let A be an $m \times n$ matrix. Suppose B and C are both $m \times n$ matrices satisfying*

$$AB = BA = I_n,$$
$$AC = CA = I_n,$$

then $B = C$.

Definition. Let A be an $n \times n$ matrix. If there exists a matrix A^{-1} satisfying

$$AA^{-1} = A^{-1}A = I_n,$$

then we call A^{-1} the **inverse** of A. We say that A is **invertible** if A^{-1} exists.

Example. Let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix}.$$

Verify that

$$A = \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

is the inverse of A.

Theorem 2. *If* A^{-1} *exists, then the* $n \times n$ *system* $A\mathbf{x} = \mathbf{b}$ *has the unique solution*

$$\mathbf{x} = A^{-1}\mathbf{b}$$

for every $\mathbf{b}$ *in* R^n*.*

Theorem 3. *An* $n \times n$ *matrix is invertible if and only if rank*$(A) = n$*.*

Corollary 4. *Let A be an $n \times n$ matrix. If $A\mathbf{x} = \mathbf{b}$ has a unique solution for some $\mathbf{b}$, then A^{-1} exists.*

Let's suppose that an $n \times n$ matrix A is invertible. This is true if and only if $\text{rank}(A) = n$, in which case A is row equivalent to the identity matrix I.

To find the inverse of A, we must solve the system $AX = I$, where X is an $n \times n$ unknown matrix. To solve this system, we make a big augmented matrix $[A|I]$ and perform Gauss-Jordan elimination. Because A is row equivalent to the identity matrix I, the augmented matrix $[A|I]$ will be row equivalent to some matrix $[I|B]$. The matrix B is the inverse of A.

This method of finding the inverse is called the **Gauss-Jordan technique**.

Example. Find A^{-1} if

$$A = \begin{bmatrix} -1 & 3 & 1 \\ 2 & -5 & 0 \\ 3 & -8 & -2 \end{bmatrix}$$

Solution.

$$
\begin{aligned}
[\mathbf{A}|\mathbf{I}] &= \left[\begin{array}{rrr|rrr} -1 & 3 & 1 & 1 & 0 & 0 \\ 2 & -5 & 0 & 0 & 1 & 0 \\ 3 & -8 & -2 & 0 & 0 & 1 \end{array}\right] \\
-R_1 &\to R_1 \left[\begin{array}{rrr|rrr} 1 & -3 & -1 & -1 & 0 & 0 \\ 2 & -5 & 0 & 0 & 1 & 0 \\ 3 & -8 & -2 & 0 & 0 & 1 \end{array}\right] \\
-2R_1 + R_2 &\to R_2 \left[\begin{array}{rrr|rrr} 1 & -3 & -1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 3 & -8 & -2 & 0 & 0 & 1 \end{array}\right] \\
-3R_1 + R_3 &\to R_3 \left[\begin{array}{rrr|rrr} 1 & -3 & -1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 1 & 1 & 3 & 0 & 1 \end{array}\right] \\
-R_2 + R_3 &\to R_3 \left[\begin{array}{rrr|rrr} 1 & -3 & -1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right] \\
3R_2 + R_1 &\to R_1 \left[\begin{array}{rrr|rrr} 1 & 0 & 5 & 5 & 3 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right] \\
-R_3 &\to R_3 \left[\begin{array}{rrr|rrr} 1 & 0 & 5 & 5 & 3 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right] \\
-5R_3 &\to R_1 \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 10 & -2 & 5 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right] \\
-2R_3 &\to R_2 \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 10 & -2 & 5 \\ 0 & 1 & 0 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right]
\end{aligned}
$$

$$
\mathbf{A}^{-1} = \left[\begin{array}{rrr} 10 & -2 & 5 \\ 4 & -1 & 2 \\ -1 & 1 & -1 \end{array}\right]
$$

CHECK:

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} -1 & 3 & 1 \\ 2 & -5 & 0 \\ 3 & -8 & -2 \end{bmatrix} \begin{bmatrix} 10 & -2 & 5 \\ 4 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} 10 & -2 & 5 \\ 4 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 1 \\ 2 & -5 & 0 \\ 3 & -8 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

□

Properties of the Inverse

Theorem 5. *Let A and B be invertible $n \times n$ matrices. Then*

1. *A^{-1} is invertible and $(A^{-1})^{-1} = A$.*
2. *AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.*
3. *A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.*

Corollary 6. *Let $A_1, A_2, \ldots, A_k$ be invertible $n\times n$ matrices. Then $A_1A_2\cdots A_k$ is invertible, and*

$$(A_1A_2\cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}\cdots A_1^{-1}.$$

Some Further Theoretical Results

Theorem 7. *Let A and B be $n \times n$ matrices. If $AB = I_n$, then both A and B are invertible and $B = A^{-1}$.*

Corollary 8. *Let A and B be $n \times n$ matrices. If AB is invertible, then both A and B are invertible.*